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ON THE DEFINITION AND CALCULATION OF EFFECTIVE IMPEDANCES . . .

Authors: A J Mackay & J G Gallagher

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RSRE MEMORANDUM 4518

Title: ON THE DEFINITION AND CALCULATION OF EFFECTIVE IMPEDANCES AND AVERAGED BOUNDARY CONDITIONS WITH APPLICATION TO THE ANALYSIS OF HIGH COMPLEXITY FREQUENCY SELECTIVE SURFACES

Authors: A.J.Mackay, J.G.Gallagher

Date: July 1991

ABSTRACT

This report addresses the use of averaged boundary conditions as a method of analysis for electrical scattering from large structures which are geometrically complex on a scale much smaller than a wavelength. The emphasis is placed on how they may be used for the study of frequency selective surfaces, and a simple example is given concerning a sparse array of small thin dipoles.

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On the definition and calculation of effective impedances and averaged boundary conditions with application to the analysis of high complexity frequency selective surfaces

A J Mackay, J.G.Gallagher

1.0 Introduction

Frequency Selective Surfaces (FSS) have a large number of applications. Examples include the design and construction of radomes and antennas and the construction of passive microwave and millimeter–wave filters. With the improvements in computer technology, numerical methods have become ever more important in FSS design. However, it still remains the case that FSS with highly complex patterns are not amenable to modal analysis methods in a realistic period of time.

In the past, approximate methods have been more frequently used than today, often because there was no realistic alternative available. Despite, and sometimes because of, modern computer abilities, it is well worth re–examining some of these methods in the light of new requirements. Indeed, such methods can often highlight aspects of the physics that may often be overlooked.

In this report, we focus our attention on the use of approximate boundary conditions and their use in the analysis of complex structures within the context of modal analysis programs for FSS design. This may be of particular importance in the study of fractal and fractal–like periodic patterns over restricted bandwidths.

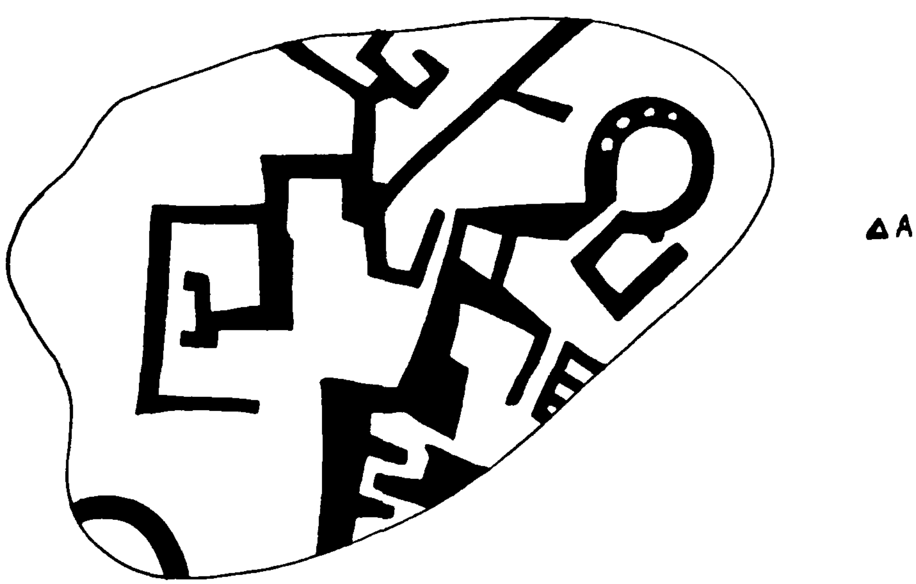
Approximate boundary conditions have been studied extensively by Kontorovich et al. (see [1]) in the early sixties and before. The motivating philosophy assumes that it is not necessary to calculate the 'exact' currents and fields induced in the neighborhood of the scatterer, on a scale very much less than a wavelength, in order to determine far fields and

transmission and reflection coefficients far from the scatterer. All that is required is to have a relationship between the averaged electric and magnetic fields on a scale possibly much larger than the fine detail of the scatterer although still small compared to a wavelength. Indeed, this is precisely what occurs with the use of natural and artificial dielectrics where fields on an atomic scale are not required for macroscopic field calculations.

1.1 Averaged Boundary Conditions

Kontorovich et al. [1] applied averaged boundary conditions to the determination of the reflection coefficient of a plane wave from a plane wire mesh. This was generalised further in [2] and extended by Astrakhan [3]. In this section, we define what we mean by averaged boundary conditions, providing a generalisation to the definition given in [2], and describe the scale relation between averaged boundary conditions and currents.

Suppose we consider the planar scattering problem, appropriate to FSS, in which there exists a complex and finely detailed structure in the plane z=0. The microstructure is taken to be an arbitrary pattern of arbitrary complex impedance (generally inhomogeneous) of zero thickness measured in ohms per square. Let us consider a region of surface small compared to a wavelength, but large compared to the scale of the detail of the structure, such that there is negligible phase variation across the area of interest. This can be represented by an area ΔA, as illustrated below, with the understanding that the boundaries are not "hard" but defined by a weight function $w_\Delta(\underline{x})$ as defined in the following section.



Within any part of the structure outlined by ΔA, the fields may be determined in terms of an incident field by quasi–static means. If the structure is split into distinct cells, each of size and shape ΔA then the i^{th} such cell, ΔA_i, is excited by a field $\underline{E}_i^{inc}(\underline{x})$ from sources outside of the cell.

In general, it is possible to define the averaged fields with respect to a weighting function, $w_\Delta(\underline{x})$, of the same functional form throughout the z=0 plane. Ignoring the i suffix, we may thus define the averaged electric fields and currents over the cell by,

$$\underline{E}^\Delta(\underline{x}) = \frac{1}{\Delta_a}\iint_U w_\Delta(\underline{x}-\underline{x}')\ \underline{E}^\delta(\underline{x}')dx' \tag{1.1a}$$

$$\underline{J}^\Delta(\underline{x}) = \frac{1}{\Delta_a}\iint_U w_\Delta(\underline{x}-\underline{x}')\ \underline{J}^\delta(\underline{x}')dx' \tag{1.1b}$$

where $\underline{E}^\delta(\underline{x})$ and $\underline{J}^\delta(\underline{x})$ are the electric fields and currents averaged over a much finer scale. If an exact point wise description of the fields and currents is defined and given by $\underline{E}(\underline{x})$ and $\underline{J}(\underline{x})$, then we may write,

$$\underline{E}^\delta(\underline{x}) = \frac{1}{\delta_a}\iint_U w_\delta(\underline{x}-\underline{x}')\ \underline{E}(\underline{x}')dx' \tag{1.1c}$$

$$\underline{J}^\delta(\underline{x}) = \frac{1}{\delta_a}\iint_U w_\delta(\underline{x}-\underline{x}')\ \underline{J}(\underline{x}')dx' \tag{1.1d}$$

where $w_\delta(x)$ represents the fine scale weight function. In general, however, $\underline{E}(x)$ and $\underline{J}(x)$ are not observables and may not be measured in any realisable experiment.

In the above definitions the domain of integration U is over the entire z=0 plane and the area functions Δ_a and δ_a are defined by

$$\Delta_a = \iint_U w_\Delta(\underline{x}')dx' \qquad (1.1e)$$

$$\delta_a = \iint_U w_\delta(\underline{x}')dx' \qquad (1.1f)$$

where the weight functions $w_\Delta(x)$ and $w_\delta(x)$ are defined to be *local*, real and non–negative. We define such a w(x) according to the rules,

(1) $w(\underline{x}) \geq 0 \quad \forall \underline{x} \epsilon U$

(2) $\iint_u w(\underline{x})dx$ exists and is finite

(3) $w(\underline{x}) \sim o\left[\frac{1}{|\underline{x}|}\right]$ as $|\underline{x}| \to \infty$

This last condition indicates that $w(\underline{x})$ is chosen to be a function that decays faster than $1/|\underline{x}|$ for $|\underline{x}|$ sufficiently large. In addition, we further restrict $w(\underline{x})$ such that

(4) $w(\underline{x})$ takes a single maximum value when $\underline{x}=0$

(5) we define the *shape boundary* as the locus $\underline{b}=\underline{b}(\theta)$, parameterised by θ, such that $|w(\underline{b})/w(\underline{0})|=0.5$ and restrict $w(\underline{x})$ so that $|\underline{b}|$ falls within the localised limits $B/2 \leq |\underline{b}(\theta)| \leq 3B/2$ for some $B \geq 0$ $\forall \theta$.

With this last definition we can specify that $b_\Delta \gg b_\delta$, associated with $w_\Delta(\underline{x})$ and $w_\delta(\underline{x})$ respectively, distinguishing between the two different length scales involved in the definitions of the averaged currents and fields. Furthermore, we can now associate Δ_a with the area of the region ΔA, and the shape of ΔA with the locus $\underline{b}(\theta)$.

We now seek to define a general tensor impedance relating $\underline{E}^\Delta(\underline{x})$ and $\underline{J}^\Delta(\underline{x})$ at the point $\underline{x}$ by,

$$\underline{E}_i^\Delta(\underline{x}) = Z_{ij}(w_\Delta(\underline{x}), \underline{x})\, \underline{J}_j^\Delta(\underline{x}) \qquad (1.2)$$

with implied summation over the index j. In this planar problem, we only need to consider field and current components within the x–y plane and $\mathbf{Z}=Z_{ij}$ may be represented as a 2×2 matrix. We note that in general $\mathbf{Z}$ will be a function both of the weight function $w_\Delta(\underline{x})$ and of the position on the plane $\underline{x}$.

Suppose the field incident on the z=0 plane is given by $\underline{E}_0(\underline{x})$, then over each ΔA_i we may assume $\underline{E}_0(\underline{x})$ is approximately constant. Let us define the tangential component of this field over the i^{th} cell as $\underline{E}_{0i}^{(tang)}$. Then we may write,

$$\underline{E}_{0i}^{(tang)} + \underline{E}_{ind}^{\delta}(\underline{x}) \approx \mathbf{Z}(w_\delta(\underline{x}), \underline{x})\, \underline{J}^{\delta}(\underline{x}) \quad \text{for } x \in \Delta A_i \cap \mathbb{D} \tag{1.3}$$

where $\mathbb{D}$ is that region of the z=0 plane where there are non–zero currents and where $\underline{E}_{ind}^{\delta}(\underline{x})$ is the tangential component of the induced electric field from currents everywhere in the z=0 plane,

$$\underline{E}_{ind}^{\delta}(\underline{x}) \approx \left[\iint_{\mathbb{D}} \mathbf{G}(\underline{x}-\underline{x}')\underline{J}^{\delta}(x')\, dx' \right]_{tang} \tag{1.4}$$

and $\mathbf{G}$ is the familiar Greens function given by

$$\mathbf{G}(\underline{x}-\underline{x}') = \frac{-j\eta}{4\pi k_0} (k_0^2\, \mathbf{I} - \nabla\nabla) \left[\frac{e^{-jk_0|\underline{x}-\underline{x}'|}}{|\underline{x}-\underline{x}'|} \right] \tag{1.5}$$

where k_0 is the free space wave number, $\mathbf{I}$ is the identity operator and η is the impedance of free space. When $\underline{x} \in \Delta A_i$ but $\underline{x} \notin \mathbb{D}$ (ie in the aperture regions), $\mathbf{Z}(w_\delta(\underline{x}), \underline{x})$ has infinite components and $\underline{J}_j^{\delta}(\underline{x})$ has zero components to match. If we are considering the vector component of (1.3) parallel to $\underline{E}_{0i}^{(tang)}$, then the right hand side of (1.3) is undefined in the aperture region. However, since (1.3) is valid for arbitrarily large but finite $\mathbf{Z}$ it is useful to

extend the region of validity with the understanding that $Z(w_\delta(\underline{x}), \underline{x})$ and $\underline{J}_j^\delta(\underline{x})$ can not be separated for $x \notin D$. Thus we now write

$$\underline{E}_{0i}^{(tang)} + \left[\iint_U G(\underline{x}-\underline{x}')\underline{J}^\delta(x')\, dx' \right]_{tang} \approx Z(w_\delta(\underline{x}), \underline{x})\, \underline{J}_j^\delta(\underline{x}) \quad \forall \underline{x} \in \Delta A_i \tag{1.6}$$

Now, returning to (1.2) in combination with (1.1a) and(1.1b), we have the expression,

$$\underline{E}_{0i}^{(tang)} + \frac{1}{\Delta_a}\iint_U w_\Delta(\underline{x}-\underline{x}')\, \underline{E}_{ind}^\delta(\underline{x}')dx' \approx Z(W_\Delta(\underline{x}), \underline{x}) . \frac{1}{\Delta_a}\iint_U w_\Delta(\underline{x}-\underline{x}')\, \underline{J}^\delta(\underline{x}')dx' \tag{1.7}$$

Substituting (1.4) in to (1.7) gives us

$$\begin{aligned} \underline{E}_{0i}^{(tang)} &+ \left[\frac{1}{\Delta_a}\iint_U w_\Delta(\underline{x}-\underline{x}') \iint_U G(\underline{x}'-\underline{x}'')\underline{J}^\delta(x'')\, dx''dx' \right]_{tang} \\ &\approx Z(w_\Delta(\underline{x}), \underline{x}) . \frac{1}{\Delta_a}\iint_U w_\Delta(\underline{x}-\underline{x}')\, \underline{J}^\delta(\underline{x}')dx' \quad \forall \underline{x} \in \Delta A_i \end{aligned} \tag{1.8}$$

We now integrate (1.6) over the entire z=0 plane with respect to the weight function $w_\Delta(\underline{x}-\underline{x}')$. Since $E_{0i}^{(tang)}$ is assumed constant over ΔA_i, we have

$$\begin{aligned} \underline{E}_{0i}^{(tang)} &+ \left[\frac{1}{\Delta_a}\iint_U w_\Delta(\underline{x}-\underline{x}'') \iint_U G(\underline{x}''-\underline{x}')\underline{J}^\delta(\underline{x}')\, dx'dx'' \right]_{tang} \\ &\approx \frac{1}{\Delta_a}\iint_U w_\Delta(\underline{x}-\underline{x}'')\, Z(w_\delta(\underline{x}''), \underline{x}'') . \underline{J}^\delta(\underline{x}'')\, dx'' \quad \forall \underline{x} \in \Delta A_i \end{aligned} \tag{1.9}$$

We may now compare (1.9) and (1.6) to show the *scale relation* between averaged currents and impedances,

$$Z(w_\Delta(\underline{x}),\underline{x}).\iint_U w_\Delta(\underline{x}-\underline{x}')\ \underline{J}^\delta(\underline{x}')dx' = \iint_U w_\Delta(\underline{x}-\underline{x}'')\ Z(w_\delta(\underline{x}''),\underline{x}'').\underline{J}^\delta(\underline{x}'')\ dx''$$

$$\forall \underline{x} \in \Delta A_i \qquad (1.10)$$

with the proviso that $Z(w_\delta(\underline{x}),\ \underline{x})\ \underline{J}^\delta(\underline{x})$ is defined everywhere according to the definition in (1.6).

1.2 The determination of $Z(w_\Delta(\underline{x}), \underline{x})$

In order to determine the averaged boundary condition $Z(w_\Delta(\underline{x}),\ \underline{x})$ it is necessary to know both the local boundary condition $Z(w_\delta(\underline{x}),\ \underline{x})$ and the local current distribution. In regions where there is an aperture and zero current, the local electric field distribution on the aperture is required. These distributions can be found by an application of the method of moments using the wave equation applied to Δ_i for each i in isolation. Alternatively, provided ΔA_i is sufficiently small, Laplace's equation may be used. Assuming, for the moment, that each cell ΔA_i contains m_i unknowns and there are n such cells (each of which is different), the full scattering problem involving all the cells can be solved by n matrix inversions each of size m_i ($1 \leq i \leq n$), followed by a single matrix inversion of size n. This may be several orders of magnitude faster than a full application of the method of moments to the union of all the Δ_i resulting in one matrix inversion of size $\sum_{i=1}^{n}(m_i)$. Furthermore, if most cells, Δ_i, are similar still greater savings will be achieved. We should note, however, that the correct determination of the current and field distributions over ΔA_i requires a larger region than ΔA_i (see below) and thus the number of unknowns will be $\beta_i m_i$ for each cell i, for some β_i possibly as large as several hundred ($\beta_i = O(a^2)$

where a is defined below). Nevertheless, provided that a solution of a problem with $\sum_{i=1}^{n}(m_i)$ unknowns is more costly than n solutions of problems with $\beta_i m_i$ unknowns, which will always be the case for sufficiently large a problem, the method is justifiable.

In order to determine what the local current distribution is we assume that $\underline{J}^{\delta}(\underline{x})$ can be found, up to an unspecified constant multiplier a, by considering the solution to

$$\underline{\hat{e}} + \iint_{\Delta A' \cap D} G(\underline{x}-\underline{x}')\underline{J}^{\delta'}(\underline{x}')\, dx' = Z(w_{\delta}(\underline{x}), \underline{x})\, \underline{J}^{\delta'}(\underline{x}) \quad \forall \underline{x} \in \Delta A'_i \tag{1.11}$$

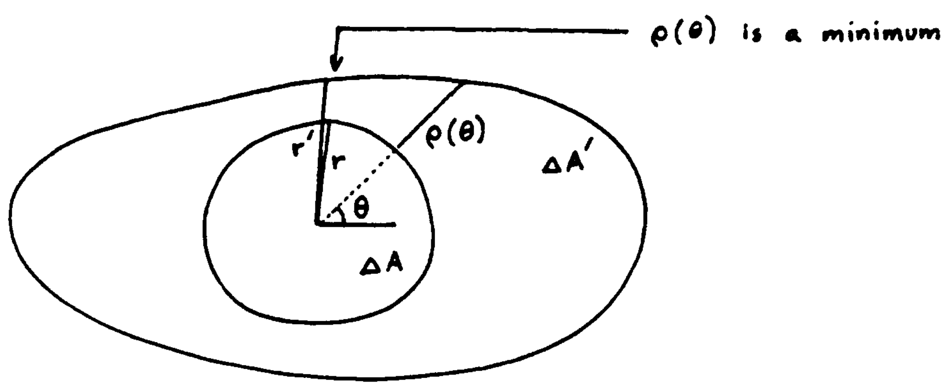
where $$\underline{J}^{\delta'}(\underline{x}) = a\underline{J}^{\delta}(\underline{x}) \quad \forall \underline{x} \in \Delta A_i,$$

and $$\underline{\hat{e}} = \underline{E}_{0i}^{(tang)} / |\underline{E}_{0i}^{(tang)}|$$

and $\Delta A'_i$ is sufficiently larger than and centred about ΔA_i. This establishes a 'guard' region where all the fields in ΔA_i induced by currents in cells outside of $\Delta A'_i$ may be taken as approximately uniform over ΔA_i. We note that this is valid only if $\Delta A'_i$ is still significantly smaller than a wavelength, in which case the Green's function may be replaced by the quasi–static approximation,

$$G(\underline{x}-\underline{x}') \approx \frac{j\eta_0}{4\pi k_0} \nabla\nabla \left[\frac{1}{|\underline{x}-\underline{x}'|} \right] \tag{1.12}$$

If the locus $\underline{b}'(\theta)$ is the shape boundary of $\Delta A'$ and $\underline{b}(\theta)$ is the shape boundary of ΔA, the radial distance between the two boundaries $\rho(\theta) = |\underline{b}'(\theta) - \underline{b}(\theta)|$. If ρ takes a minimum value ρ_{min} at $\theta = \theta_{min}$ then we define $r' = |\underline{b}'(\theta_{min})|$ and $r = |\underline{b}(\theta_{min})|$ and the size of the guard region may be determined by the ratio $a = r'/r \geq 1$. This is illustrated in the figure below,



In theory, the minimum a should be determined such that any (quasi–static) dipole source situated at the point $\underline{x}_0$ (with a $1/|\underline{x}-\underline{x}_0|^2$ electric field dependence) anywhere on the boundary of $\Delta A'$ causes a negligible difference in field strength over any part of the boundary of ΔA. Given this criterion and defining ξ as the maximum permitted fractional difference in field strength between the centre of ΔA (at $\underline{x}=0$) and its boundary, then we must satisfy the inequality,

$$\frac{1}{(a-1)^2} - \frac{1}{a^2} \leq \frac{\xi}{a^2} \tag{1.13}$$

which, for small ξ, gives rise to the condition that $a \gtrsim 2/\xi$.

If we allow a reasonable choice of ξ=0.1, it is seen that $a \approx 20$ which is a figure that seems rather high. In practice, it may often be possible to significantly reduce a below this limit. This reduction is clearly important for any numerical solution of (1.11).

2.0 Example of the method

In order to implement the method it is first necessary to choose suitable weight functions. For purposes of illustration, we choose a square pulse function of width ΔX and area $\Delta A_a = \Delta^2$,

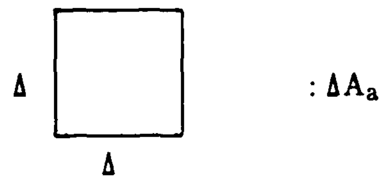


Fig 2.1

We now seek to apply the use of approximate boundary conditions to a FSS which is amenable to simple analysis. In particular, we consider a sparse array of electrically small dipoles on a square lattice. Part of such an array is illustrated below,

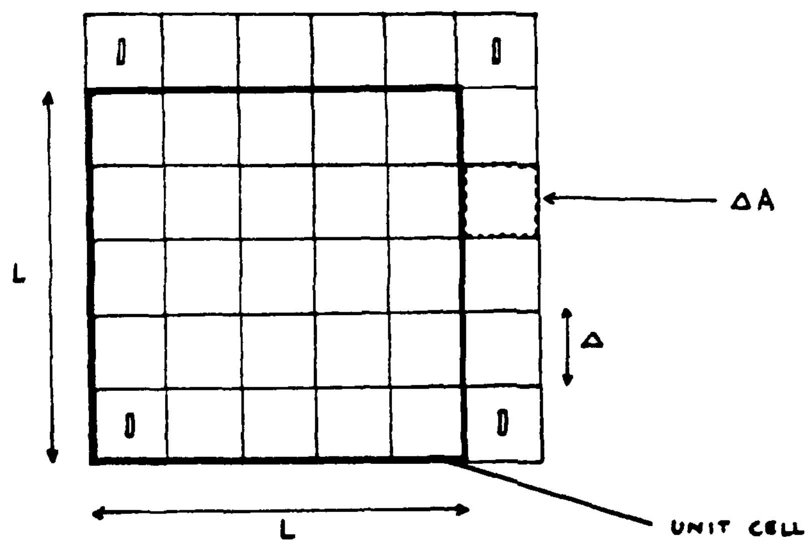


Fig 2.2

In this representation we choose a grid such that each unit cell of the FSS is divided into $n \times n$ equally sized squares, one of which includes a small dipole at its centre. The unit cell is square and of size L such that $L = n\Delta$. Suppose that we identify each such cell with ΔA and wish to find $Z(w_\Delta(\underline{x}), \underline{x}_i)$ at each of the i $(1 \leq i \leq n^2)$ cells, where $w_\Delta(\underline{x})$ is the pulse function

associated with ΔA, defined by

$$w_{\Delta}(\underline{x}) = \begin{cases} 1 & \underline{x} \in \Delta A \\ 0 & \underline{x} \notin \Delta A \end{cases} \tag{2.1}$$

Assuming that the dipoles are sparse, we may take $\Delta A'$ as the unit cell or any smaller cell since there are no current sources other than at the small dipoles. In particular, we may define $\Delta A' = \Delta A$ and $\underline{J}^{\delta'}(\underline{x}) = \underline{J}^{\delta}(\underline{x})$ for $\underline{x} \in \Delta A$.

For a small thin dipole of length 2h, radius a, situated at the origin and aligned with the y–axis, the current per unit length[1] [3] is given by,

$$I_y \approx I_0 \frac{(\cos k_0 y - \cos k_0 h)}{1 - \cos k_0 h} \tag{2.2a}$$

where

$$I_0 \approx \frac{jE_0^{par}}{30 k_0 \Omega} \frac{(1 - \cos k_0 h)}{\cos k_0 h} \tag{2.2b}$$

and

$$\Omega = 2\log_e\left[\frac{2h}{a}\right] \qquad \text{for } a \ll h \tag{2.2c}$$

and where E_0^{par} is the component of the incident electric field in the y–direction. In the case where $k_0 h \ll 1$, this may be further approximated to give the small dipole result,

$$I_y(y) \approx I_0 (h^2 - y^2)/h^2 \tag{2.3}$$

where

$$I_0 = \frac{jE_0^{par}}{30\Omega} \frac{k_0 h^2}{2}$$

If we assume an electric field incident parallel to the dipole, we define $\underline{J}_0 = \underline{J}^{\Delta}(\underline{0})$ and

---

[1] note that this is not the current density per unit area.

$\underline{E}_0=\underline{E}^{\Delta}(\underline{0})$. Using the definition (1.1b) with $\underline{J}^{\delta}(\underline{x})=\underline{J}(\underline{x})$, we can see that

$$\underline{J}^{\Delta}(\underline{x}) \approx \begin{cases} \underline{J}_0 & \forall\, \underline{x}\in \Delta A \\ 0 & \forall\, \underline{x}\notin \Delta A \end{cases} \text{, when } h \ll \Delta/2$$

where

$$\underline{J}_0 = \iint_{\Delta A} \underline{J}(\underline{r})\, da = \hat{\underline{y}} \int_{-h}^{h} \frac{I_0}{h^2} (h^2 - y^2)\, dy = \frac{4}{3} I_0 h \hat{\underline{y}} \quad , k_0 h \to 0 \tag{2.4}$$

for incidence perpendicular to the dipole, this integral is zero. For parallel incidence the integrated electric field over ΔA, $\underline{E}^{\Delta}(\underline{x})$ does *not* take an approximately constant value for $\underline{x}\in\Delta A$. In particular, $\underline{E}^{\Delta}(\underline{x})\neq\underline{E}_0$ for $\underline{x}\in\Delta A$ when $h\ll\Delta/2$. Using the expression for $G(x-x')$ given in (1.12), with the quasi–static simplification,

$$\nabla\nabla\left[\frac{1}{|\underline{x}-\underline{x}'|}\right]\cdot\hat{\underline{y}}\bigg|_{x'=0} = \frac{3x(y-y')}{[x^2+(y-y')^2]^{5/2}}\,\hat{\underline{x}} + \frac{2(y-y')^2 - x^2}{[x^2+(y-y')^2]^{5/2}}\,\hat{\underline{y}}$$

and the definition (1.1a) with $\underline{E}^{\delta}(\underline{x})=\underline{E}(\underline{x})$ we can see that if we assume negligable width to the dipole,

$$\underline{E}^{\Delta}(\underline{x}) = \frac{-j\eta_0}{4\pi k_0}\frac{I_0}{h^2}\left[\mathcal{J}_1^{(y)}(\underline{x})\hat{\underline{y}} + \mathcal{J}_1^{(x)}(\underline{x})\hat{\underline{x}}\right] \tag{2.5}$$

where

$$\mathcal{J}_1^{(y)}(\underline{x}) = \int_{-\Delta/2}^{\Delta/2} dy'' \int_{-h}^{h} dy'\,(h^2-y'^2) \int_{-\Delta/2}^{\Delta/2} dx'' \frac{2(y''+y-y')^2 - (x''+x)^2}{[(x''+x)^2+(y''+y-y')^2]^{5/2}}$$

and

$$\mathfrak{I}_1^{(x)}(\underline{x}) = \int_{-\Delta/2}^{\Delta/2} dy'' \int_{-h}^{h} dy'(h^2-y'^2) \int_{-\Delta/2}^{\Delta/2} dx'' \frac{3(y''+y-y')(x''+x)}{[(x''+x)^2+(y''+y-y')^2]^{5/2}}$$

Focusing our attention on the $\hat{y}$ component of (2.5), since we do not excite the dipole in the $\hat{x}$ direction, we may make use of standard integrals to show that

$$\mathfrak{I}_1^{(y)}(\underline{x}) = \mathfrak{I}_2(\eta^+,\xi^+)-\mathfrak{I}_2(\eta^+,\xi^-)-\mathfrak{I}_2(\eta^-,\xi^+)+\mathfrak{I}_2(\eta^-,\xi^-) \tag{2.6}$$

where

$$\eta^+ = y+\Delta/2 \tag{2.7a}$$

$$\eta^- = y-\Delta/2 \tag{2.7b}$$

$$\xi^+ = x+\Delta/2 \tag{2.7c}$$

$$\xi^- = x-\Delta/2 \tag{2.7d}$$

and where the function $\mathfrak{I}_2(\eta,\xi)$ is defined by,

$$\begin{aligned}\mathfrak{I}_2(\eta,\xi) = {} & (h^2-\eta^2)\log_e\left|\frac{(\sqrt{[\xi^2+(\eta-h)^2]}-\xi)(\sqrt{[\xi^2+(\eta+h)^2]}+\xi)}{\eta^2-h^2}\right| \\ & + 2\eta\xi\log_e\left|\frac{(\sqrt{[\xi^2+(\eta-h)^2]}+\eta-h)(\sqrt{[\xi^2+(\eta+h)^2]}-\eta-h)}{\xi^2}\right| \\ & + \xi\left[-\sqrt{[\xi^2+(\eta-h)^2]}+\sqrt{[\xi^2+(\eta+h)^2]}\right]\end{aligned} \tag{2.8}$$

In order to illustrate $\mathfrak{I}_1^{(y)}(\underline{x})$, in figures 2.3a,b,c we produce a contour plot of this function for $h=0.1\Delta$, $h=0.2\Delta$ and $h=0.3\Delta$ respectively. $\mathfrak{I}_1^{(y)}(x,y)$ is plotted for $-\Delta/2 \leq x \leq \Delta/2$ and $-\Delta/2 \leq y \leq \Delta/2$ and Δ is taken as unity ($\Delta=1$). It is immediately apparent that there is a significant variation in averaged field strength over the area ΔA with respect to the y coordinate. The variation in the x direction is much less significant. However, it is clear

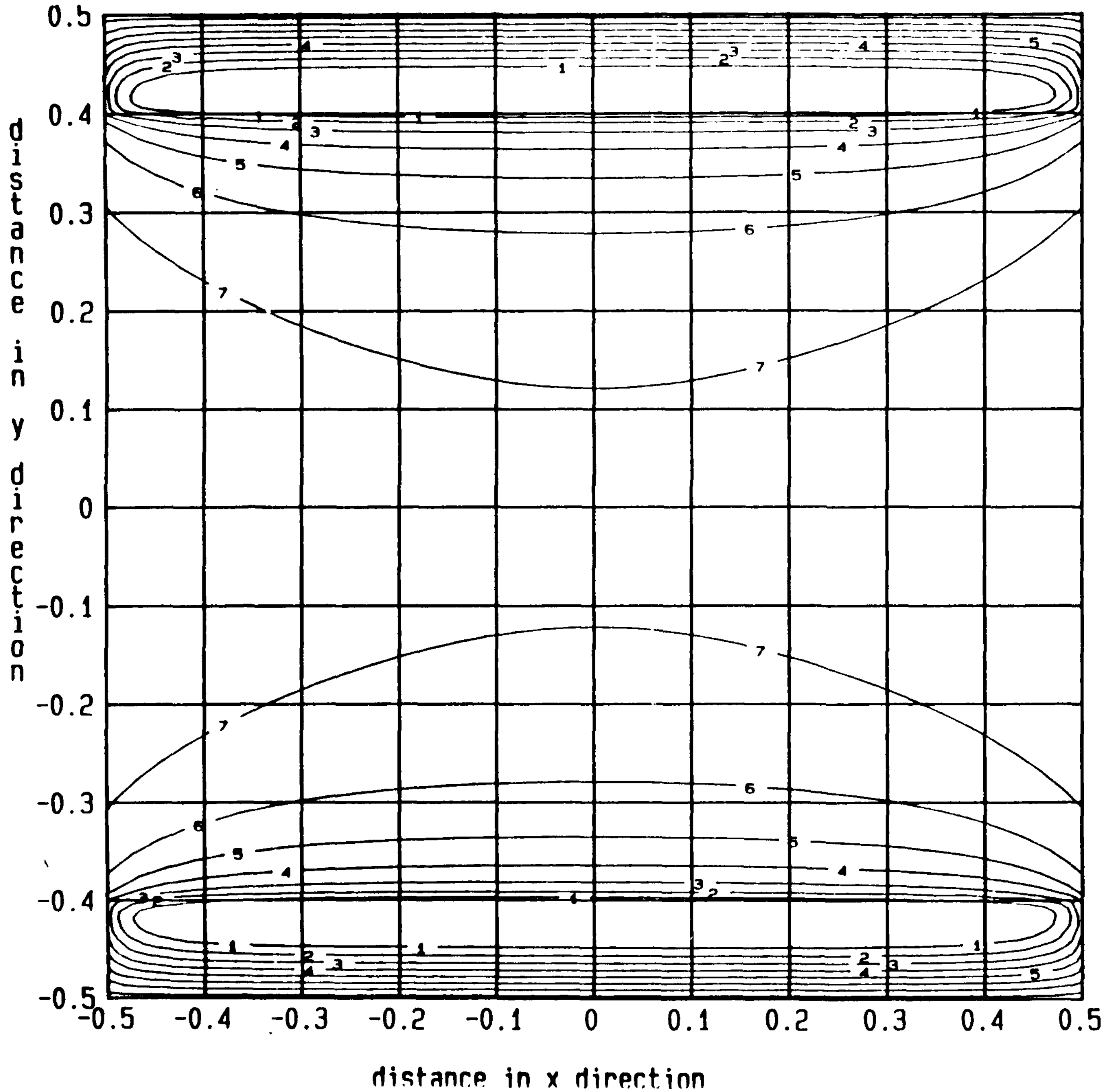
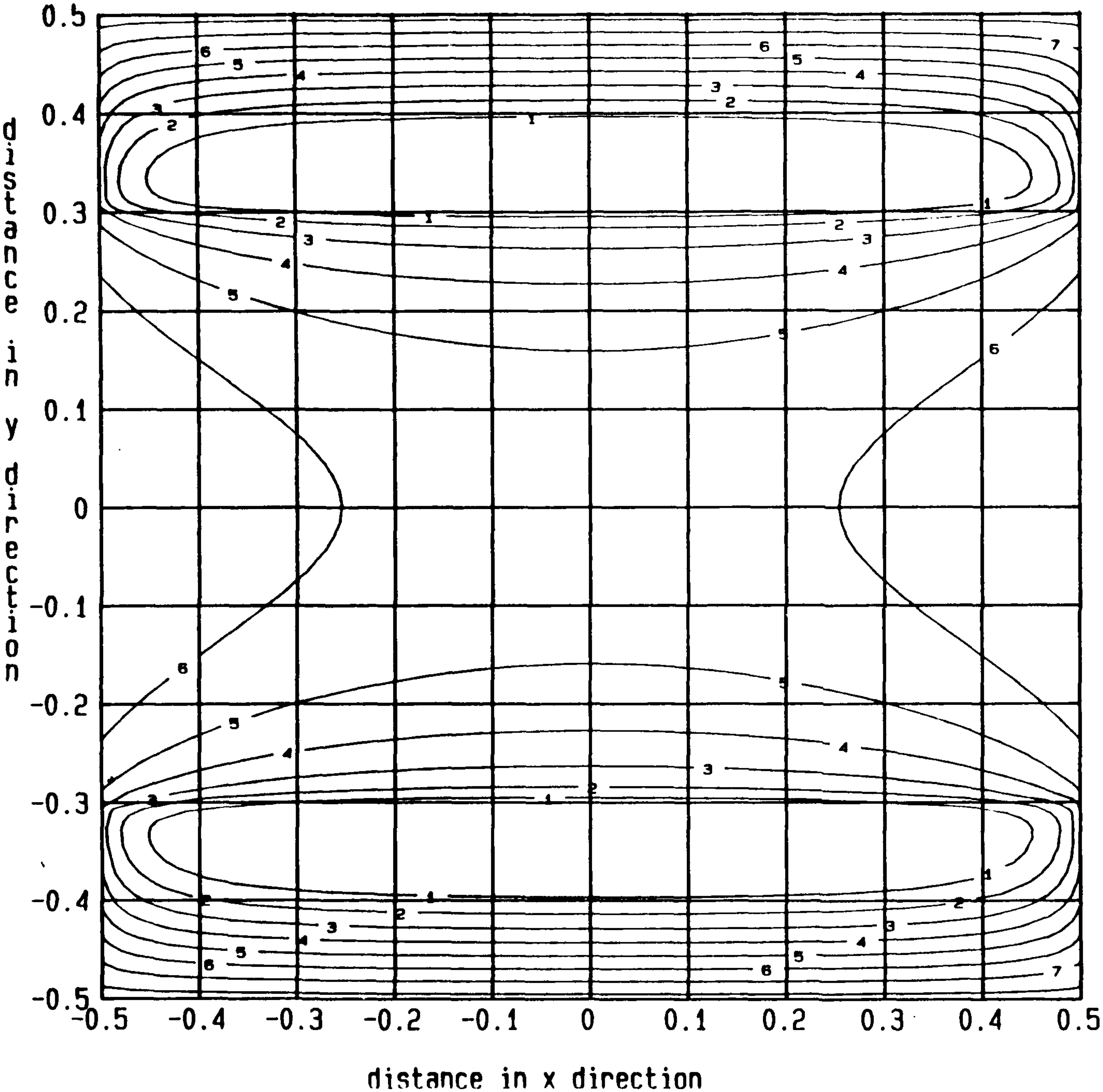


Figure 2.3a **h=0.1Å**

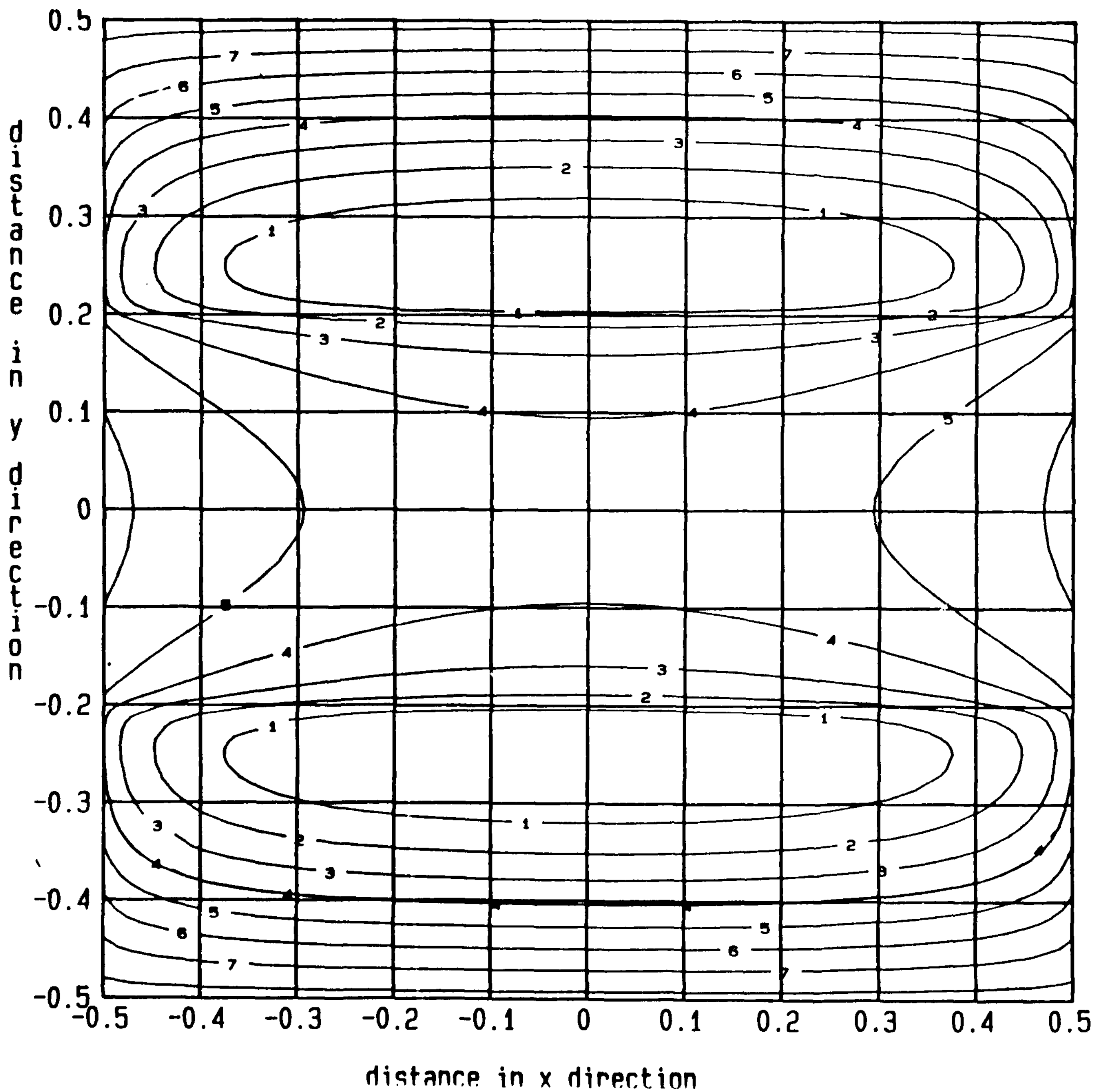
CONTOUR KEY	
1	-0.0384
2	-0.0334
3	-0.0284
4	-0.0234
5	-0.0184
6	-0.0134
7	-0.0084
8	-0.0034

Figure 2.3b h=0.2Å



CONTOUR KEY	
1	-0.1573
2	-0.1373
3	-0.1174
4	-0.0974
5	-0.0775
6	-0.0575
7	-0.0376
8	-0.0176

Figure 2.3c h=0.3Å



CONTOUR KEY	
1	-0.4035
2	-0.3532
3	-0.3029
4	-0.2526
5	-0.2023
6	-0.1520
7	-0.1017
8	-0.0514

that for $h \ll \Delta$, $\mathcal{J}_1^{(y)}(\underline{x})/\Delta^2 \to 0 \ \forall \underline{x}$. This is merely saying that the averaged scattered electric field from a small dipole becomes smaller as the dipole size is reduced. It is, for example, relatively easy to show that,

$$\underline{E}_0 = \hat{\underline{y}}\,(j\eta_0/\pi k_0) I_0\,(2h/\Delta) \quad \text{as } h/\Delta \to 0. \tag{2.9}$$

We may now use (1.10) to define the terms of the effective impedance matrix $\mathbf{Z}(w_\Delta(\underline{x}),\underline{x})$ for all $\underline{x}$ within the cell ΔA containing the dipole[2], $\mathbf{Z} = \begin{bmatrix} \infty & \infty \\ \infty & Z_{yy} \end{bmatrix}$, where Z_{yy} is given by

$$Z_{yy}(\underline{x}) = \frac{\iint_{\Delta A} E_0^{par} da \; + \underline{E}^{\Delta}(\underline{x}) \cdot \hat{\underline{y}}}{\underline{J}_{avg} \cdot \hat{\underline{y}}} \qquad \forall \underline{x} \in \Delta A$$

Assuming $\underline{E}^{\Delta}(\underline{x})$ is sufficiently small, as discussed above for small dipoles, we may use (2.4) and (2.2) to obtain the expression,

$$Z_{yy} \approx -j\,\frac{45\Omega}{k_0 h}\left[\frac{\Delta}{h}\right]^2 \quad \text{for } h \ll \Delta/2,\ k_0 h \ll 1 \tag{2.11}$$

where Z_{yy} is independent of $\underline{x}$ for $\underline{x} \in \Delta A$. We note once again that this effective impedance is approximately correct provided we *define* the weight function $w_\Delta(\underline{x})$ to be the pulse basis function given by (2.1) and thus take the current distribution to be approximately uniform. Such an averaged current distribution coupled with the effective impedance must automatically be consistent with Maxwell's equations. This is confirmed in this example, since a small area of material with high self impedance (note that Z_{yy} is large and capacitive when $h \ll \Delta$ and $k_0 h \ll 1$) is known to exhibit an induced current distribution that is

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[2] all the terms of $\mathbf{Z}$ for cells ΔA which do not contain a dipole are infinite.

uniform over most of the material.

In order to check the validity of this expression, we compare the far field vector potential generated by the induced current in the dipole with the effective impedance method applied to the uniform impedance square.

The scattered far field generated by an electrically small dipole with negligible width is defined by a vector potential $\underline{A}_{dip}(\underline{x})$. Using the expression in Jackson [4], the far field vector potential is given by

$$\lim_{r\to\infty} \underline{A}_{dip}(\underline{x}) = \frac{e^{-jkr}}{4\pi r}\int_{-h}^{h} \underline{J}_{dip}(\underline{x}')dx'$$

where $\underline{J}_{dip}(\underline{x})$ is the current distribution on the dipole. This integral has already been evaluated in (2.4), thus

$$\lim_{r\to\infty} \underline{A}_{dip}(\underline{x}) = \frac{e^{-jkr}}{4\pi r}\frac{4}{3} I_0 h\hat{\underline{y}} = \frac{e^{-jkr}}{4\pi r}\frac{1}{45}\frac{jE_0^{par}}{\Pi} k_0 h^3 \tag{2.12}$$

Similarly, the scattered far field from the small impedance patch with constant surface impedance Z_{yy} and pulse current distribution, $\underline{J}_{eff}=J_0\hat{\underline{y}}$, gives rise to a far–field vector potential,

$$\lim_{r\to\infty} \underline{A}_{eff}(\underline{x}) = \frac{e^{-jkr}}{4\pi r}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}\int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} dx\, dy\, \underline{J}_{eff}(\underline{x}')dx'$$

i.e.

$$\lim_{r\to\infty} \underline{A}_{eff}(\underline{x}) = \frac{e^{-jkr}}{4\pi r}\Delta^2 J_0\hat{\underline{y}} \tag{2.13}$$

but the impedance boundary condition states that

$$J_0 \approx E_0^{ar}/Z_{yy}$$

assuming that the scattered electric field over the small impedance square is small compared with the incident field strength E_0^{ar}. Substituting the expression (2.11) into (2.13) shows that

$$\lim_{r \to \infty} \underline{A}_{eff}(\underline{x}) = \frac{e^{-jkr}}{4\pi r} \frac{1}{45} \frac{jE_0^{ar}}{\Pi} k_0 h^3 \qquad (2.14)$$

which is the same as (2.12), confirming the comparison. We would also note that in this example we assume that the scattered field is much smaller than the incident field and thus an FSS composed of such dipoles or impedance squares suffers negligible mutual interactions. Therefore, we conclude that the properties of the FSS arrays of dipoles and the effective impedance squares must also be identical.

3.0 Conclusion

Although we have not presented an example showing a use of the theory of averaged boundary conditions for a useful FSS, we have illustrated that the method is valid and have developed a generalisation of the theory over existing work [1–3]. It is hoped that the work presented here provides a basis for the development of predictive software tools to allow the analysis of structures which are currently too complex to be examined using conventional modal analysis methods.

4.0 References

[1] "The coefficient of reflection of a plane electromagnetic wave from a plane wire mesh" M.I.Kontorovich, V.Yu.Petrun'kin, N.A.Yesepkina, M.I.Astrakhan, Radio Engineering and Electronic Physics, 1962, Vol 7, No 2, pp222–231

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[4] "Classical Electrodynamics", J.D.Jackson, John Wiley & Sons 1975, sect 9.2, pp393–394

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Abstract

This report addresses the use of averaged boundary conditions as a method of analysis for electrical scattering from large structures which are geometrically complex on a scale much smaller than a wavelength. The emphasis is placed on how they may be used for the study of frequency selective surfaces, and a simple example is given concerning a sparse array of small thin dipoles.

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